# Linear Equations

## Linear Combination

a, b, c ∈ scalar

v, w, u ∈ vector

av+ bw+ cu = {[av1+bw1+cu1],

[av2+bw2+cu2]}

## Linear Equations

ex.1

x - 2y = 1

3x + 2y = 11

=

Ax = b

### row picture:

找交點

### column picture:

= → =

[1;3] & [-2;2] 做線性組合後成為[1;11]

找線性組合

當b改變時，線段改變了

x - 2y = 1 ≠ x - 2y = 2

row picture 改變了

column picture 仍然是找[1;3] & [-2;2] 的線性組合

A中column不在同一線上，b可為任一向量

## Gaussian Elimination

2x + 4y - 2z = 2

4x + 9y -3z = 8

-2x – 3y + 7z = 10

如何有系統的消?

把跟pivot 同樣變量的係數消成0

→形成一個上三角矩陣

→ →

pivot = 0 的狀況

ex: → singular case 可能有解可能無解

## Matrix Operations

### 基本算法

=

c1 =

=

r3 = a3 [1 2 4] + b3 [-2 3 1] + c3[-4 1 2]

都是做linear combination

### 運算規則

A, B, C ∈ matrix

A(BC) = (AB)C → Associative law (結合率) do hold

A(B+C) = AB + AC

(B+C)A = BA + CA → distributive law (分配率) do hold

→ 如何描述→所做的事?

可視為 =

=

3x3 3x4 3x4

原r2 -2\*r1 = 後r2 ; 原r3 +1r1 = 後r3

elementary matrix

→

可視為 =

原r2\*0 + r3 = r3 ; 原r3\*0 +r2 = r2

permutation matrix

越新的操作，乘在越左邊

## Inverse Matrix

Definition: inverse matrix

A ∈ R n x n 可逆

AB = BA = I

B = A-1

CLAIM:

Suppose A is invertible, A-1 is unique.

proof:(證明唯一：假設兩個)

如果A有兩個invers matrices B &C

BA = I & AC = I

B = BI = B(AC) = (BA)C = IC = C

Remark:

IF BA = AC = I, B=C=A-1 → right inverse = left inverse = inverse

CLAIM:

The inverse of A-1 is A.

CLAIM:

If A is invertible the only one solution of Ax=b, x=A-1b.

*proof:*

Ax = b

A-1Ax = A-1b

Ix = x = A-1b

CLAIM:

Suppose there is a nonzero solution, x to Ax=0 then A can’t be invertible. (Ax=0, 齊性解)

*proof:* (反證)

If A is invertible

Ax = 0

A-1Ax = A-10

x = 0

x 的解一定是0，有非0解一定不是invertible.

CLAIM:

A diagonal matrix has an inverse provided no diagonal entries are zero.

*proof:*

If A = A-1 =

CLAIM:

If A, B are invertible, (AB) is invertible (AB)-1=B-1A-1.

*proof:*

(AB)(B-1A-1) = A(BB-1)A-1 = AIA-1 = AA-1 = I

(B-1A-1)(AB) = B-1(AA-1)B = B-1IB = B-1B = I

CLAIM:

(ABC)-1=C-1B-1A-1

Example:

A = A-1 =

elementary matrix 的反矩陣是在非對角線上的元素上加一個負號

B = B-1 =

permutation matrix 的反矩陣是他自己

## Gauss-Jordan Elimination

Give A, find A-1

A-1A=I

3x3 ex:

A = =

Ax1 = e1 ; Ax2 = e2 ; Ax3 = e3

→可視為解3個聯立方程式

x1 = x2 = x3 =

A-1 =

→ 如果nxn矩陣可以找到 n 個pivot，此矩陣為nonsingular

CLAIM:

A matrix is invertible if and only if it’s nonsingular.

proof: (if and only if 正反都要推)

(反推)

Suppose A is nxn nonsingular, we can find a matrix B by G-J elimination, such that

AB = I ……B right inverse

(nonsingular 一定可以找出Axi = I 有解, i =1, 2, 3…n)

G-J elimination, 其實就是將A 乘上elementary matrix [E], permutation matrix [P] 最後乘上對角矩陣的反矩陣D-1

(D-1E…P…E)A = I = CA ......C left inverse

where

E = 新ri = 舊ri + x倍的rj

P = 把行互換

D-1 = 把對角線變成1

根據定義 B = C = A-1

(正推) A是singular, A invertible

If A is singular, A沒有n個pivot, G-J 最後有 全0 row

→假設會有一個可逆矩陣M(E, P, D可逆，所以M可逆) 將MA有全0 row

If A is invertible, AB = I is possible,

MAB = MI = M

(MA) 有全0 row → M有全0 row

But

如果M有全0 row 不可能invertible (不存在任意矩陣X 能夠將 MX = I)，但M invertible 是確定的(because E, P, D可逆)

所以

(MA) 有全0 row → M有全0 row不成立

→MAB = MI = M不成立

→AB = I 不成立， A is not invertible

In another viewpoint

→

A可以變成I，有足夠的pivot，A可逆

because

A-1 =

## Elimination = Factorization A = LU

→ → →

=

( E32 E31 E21 ) A = U (upper triangular matrix)

because E32, E31, E21 invertible

A = (E21-1 E31-1 E32-1) U

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| original | 此例 | general | inverse | 此例 | general |
| E21 |  |  | E21-1 |  |  |
| E31 |  |  | E31-1 |  |  |
| E32 |  |  | E32-1 |  |  |
| (E32 E31 E21) |  |  | (E21-1 E31-1 E32-1) |  |  |

lower triangle matrix (L)

A = L U =

NO row exchange

=

L 代表高斯消去法的過程(-*l* )

U代表高斯消去法的結果

Further split U

=

U = D U

A = L D U If no row exchange are required

L = lower-triangle matrix with 1 on diagonal. (record the steps elimination)

D = Diagonal matrix with pivots(nonzero) on diagonal.

U = upper-triangle matrix with 1 on diagonal.(result of the elimination)

CLAIM:

If A = L1 D1 U1 and A = L2 D2 U2 ,

where

L = lower-triangle matrix with 1 on diagonal.

D = Diagonal matrix with pivots(nonzero) on diagonal.

U = upper-triangle matrix with 1 on diagonal.

L1 = L2 , D1 = D2, U1 = U2 (此分解是唯一的)

proof:

## Transposes and Permutations

### Transposes

A = AT =

Aij = (AT)ji

CLAIM:

(A + B)T = AT + BT

CLAIM:

(AB)T = BTAT

Remark:

(ABC)T = CT BTAT

CLAIM:

(A-1)T = (AT)-1

proof:

AA-1 = I → (A-1)TAT = I

A-1A = I → AT(A-1)T = I

(A-1)T = (AT)-1

Anxn is symmetric if AT = A (對稱)

Remark:

symmetric Aij = Aji

CLAIM:

Given any matrix R, RTR & RRT is symmetric

proof:

(RTR)T = RT(RT)T = RTR

(RRT)T = (RT)TRT = RRT

A is symmetric

A = =

A = L D U = L D L-1

CLAIM:

Symmetric matrix is factored into LDU without row exchange, U = L-1

proof:

A = LDU

A = AT = UTDTLT = UTDLT = LDU

因為A = LDU is unique

所以UT = L, LT = U

### Permutations

Definition: permutation matrix

Permutation matrix is an identity matrix 的row隨便排列組合

I =

3x3 有 6種Permutation matrix

nxn 有n!種Permutation matrix

CLAIM:

If P is a Permutation matrix, P-1 = PT

proof:

P =, PT =

(P PT)ii = 1; (P PT)ij = 0

P PT = I , P-1 = PT

If row exchanges are needed

(E…P…E)A = U → A = (E-1…P-1…E-1)U

1.一開始就把row排好，乘上一Permutation matrix(P) PA，之後只做高斯消去

PA = LU

2. 高斯消去做完再排row，

L-1A = P U’; A = L P U’

# Vector Spaces and Subspaces

## Definition and Example

Rn = all vectors(column) with n real compoment

= {(v1,v2,…,vn)}: vi ∈ R ;i =1,2,…,n

∈ R2 ; (0,1,2,3,4) ∈ R5

vector space: V

Definition: vector space

* V is a set of vectors
* All vectors ∈ V
* c1v1+c2v2+c3v3+…+cnvn ∈ V
* F(field) : 表示所有scalar cn所在的集合

Rules:

1. v1 + v2 = v2 + v1
2. v1 + (v2 + v3) = (v1 + v2) + v3
3. 必須包含“zero vector”: v + 0 = v
4. For each v, there is a unique vector –v, such v + (-v) = 0
5. 1 \* v = v
6. (c1\*c2)\*v = c1\*(c2\*v)
7. c\*(v + w) = cv + cw
8. (c1 + c2)\*v = c1v + c2v

***Example***s of vector space:

1. Rn
2. M = {all real 2x2 matrices} ….把v用2x2代進去
3. G = {all real functions} …. 把v用f(x)代進去

f(x) + g(x) ∈ G

cf(x) ∈ G

1. Z = {0} (0 ∈ Rn)

Remark:

→ 0\*v = 0

Remark:

0 = 0\*v = (1+(-1))v = v + (-1)v → (-1)v = -v

Subspace

Definition: subspace

A subset W of a vector space, where W is also a vector space, W is a subspace.

其實只需要確認兩點

1. v, w ∈ W, v + w ∈ W
2. v ∈ W, cv ∈ W for any c

(這兩點都滿足→W是封閉性的)

Examples of subspace:

1. U = {(x, y) x, y ≥ 0} ≠ subspace

-1(1, 1) = (-1, -1) 不屬於 U

1. M = {: a, b, c, d ∈ R}

U = {: a, b, c, d ∈ R}

U 是 M 的subspace

## Column Space

Definition: column space

The column space C(A) of a matrix A consists of all linear combinations of the columns of A.

Example of column space:

1. A =

C(A) = { : x1,x2 ∈ R}

= {Ax̅ : x̅ ∈ ∈ R2}

若Ax̅ = b 有解，b ∈ C(A) (b 在A的column space裡)

1. I = A = B =

C(I) =

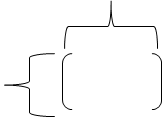
C(A) =

C(B) =

→because any b1, b2 has solution, C(B) = R2.

CLAIM:

If A ∈ Rmxn matrix, C(A) is a subspace of Rm.



m

n

S = a set of vectors in a vector space V (有可能不是subspace)

SS = the set of all linear combination of vectors in S, SS is a subspace of V

(Span of S ; the subspace spaned by S).

## Nullspace of A

Definition: nullspace

Nullspace of A(kernel of A) = all solution of Ax=0. (所有齊性解)

N(A) = {x : Ax = 0}

CLAIM:

Amxn , N(A) is subspace of Rn.

Ax = 0 A = mxn; x = nx1 → Rn

proof:

* If Ax = 0, Ay = 0, then A(x+y) = 0+0 = 0
* If Ax = 0, then A(cx) = c(Ax) = c0 = 0

Example of Nullspace:

1. A = find N(A)

A =

N(A) = x1 + 2\*x2 = 0

1. B = find N(B)

B =

N(B) = x1 = 0, x2 = 0

1. C =

C =

N(C) = x1 = 0, x2 = 0

1. D =

D =

N(D) =

→ reduce row echelon form (有pivot的該column都是0, pivot = 1)

聯立方程式： x1 + 2\*x3 = 0, x2 + 2\*x4 = 0 →pivot variable, free variable

N(D) = {x: x = ; x3, x4 ∈ R}

是多少個向量的線性組合? 由free variable有幾個決定

free variable有幾個? 由total var – pivot var = free var決定

將聯立方程式中的free variable代1其他代0，可得相對的向量

如何解 Ax̅ = 0

A

→ U (Gaussian)

→ R (reduce row echelon form, )

solve Rx̅ = 0

***Example*** of Ax̅ = 0̅

1. A =

→ U =

→ R =

Rx̅ = = 0 ; x1 + x2 + x4 = 0, x3 + x4 = 0

x̅ = (代free var x2=1, x4 =0; x2=0, x4=1)

N(A) = {x: x̅ = ; x2,x4 ∈ R }

General:

A = m x n

number of pivot r ≤ m , r ≤ n (線性獨立行個數)

number of var = n (column個數)

number of free var = n – r (線性相依行個數)

number of special solution = n – r

N(A) = span of special solutions.

1. A = mxn, n > m (columns > rows, 未知數比方程式多)

r ≤ m < n , n - r must > 0 , must have nonzero silution

Definition: rank

number of pivot r = rank of a matrix

## Pivot Columns and Special Solutions

A =

→ R = pivot column, free column

rank = 2, n – r = 3

x̅ =

N(A) = {x: x̅ = ; x2,x4,x5 ∈ R }

Ax̅ = 0 ↔ Rx̅ = 0

A= 0 ↔ R= 0 ｜-3[a1]+[a2]=0 →[a2]=3[a1]

A= 0 ↔ R= 0 ｜[a4]=2[a1]+4[a3]

A= 0 ↔ R= 0 ｜[a5]=-1[a1]-3[a3], free column, pivot column

Note:

1. Every free column is a linear combination of pivot columns
2. Special solutions describe these combinations.
3. R = =

Nullspace matrix N = =

## Complete solution of Ax̅ = b̅

Ax̅ = b̅

x̅ = = ; x2,x4 ∈ R

= homogeneous solution x̅n

= particular solution x̅p

x̅ = x̅p + x̅n

CLAIM:

If Ax̅ = b̅, the complete solution x̅ = x̅p + x̅n, where x̅p is particular solution to Ax̅ = b̅, x̅n is the general solution to Ax̅ = 0̅

proof:

* If x̅ = x̅p + x̅n

Ax̅ = A(x̅p + x̅n) = Ax̅p + Ax̅n = b̅ +0̅ = b̅

x̅ is a solution to Ax̅ = b̅

* If x̅ is a solution to Ax̅ = b̅

Ax̅ = A(x̅ + x̅p) = Ax̅ + Ax̅p = b̅ - b̅ = 0̅

x̅ - x̅p is a solution to Ax̅ = 0̅

x̅ - x̅p = x̅n → x̅ = x̅p + x̅n

Note: x̅ has solution only when 0 row rn with the associate value is 0

To find the particular solution → set all free var = 0.

To find the general solution → set a free var = 1, others = 0 respectively.

x̅n = =

x̅ = x̅p + x̅n =

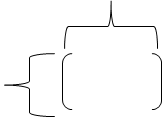
A = m x n matrix

1. If rank = n (m must ≥ n), A has full column rank

A = → R =

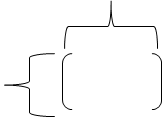
I

0



m

n



m

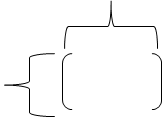
n

* 1. All columns are pivot columns
  2. A doesn’t have free var → no special solution
  3. N(A) = { 0 } (N(A) is the description of special solutions linear combination. )
  4. If Ax̅ = b̅ has solution, the solution is unique.

1. IF rank = m (m must ≤ n) , A has full row rank

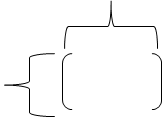
A =→ R =

I F



m

n



m

n

* 1. All rows have pivots, no 0 row.
  2. Ax̅ = b̅ always have solution. (one or infinity)
  3. C(A) = Rm
  4. number of free var = n – r = n – m = number of special solutions in N(A).

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| case | rank = m = n  (A invertible) | rank = m < n  (full row rank) | rank = n < m  (full column rank) | rank < m,  rank < n  A not full rank |
|  | R = I  Ax̅ = b̅ has 1 solution:  x̅ = A-1b̅. | R = [I F]  Ax̅ = b̅ has infinite solution. | R =  Ax̅ = b̅ has  0 or 1 solution. | R =  Ax̅ = b̅ has  0 or infinite solutions. |

## Independence, Basis and Dimension

### Independence & dependence

Definition: independence

If x1v1 + x2v2 +…+ xnvn = 0 only happens when x1=x2=…=0, vn are linearly independent (or just independent). Otherwise linearly dependent.

Example of independent:

1. → → x1 = 0, x2 = 0 → independent.

Remark:

Special solutions in nullspace of a matrix are independent.

1. → x1 →

求R, , x̅ = x3

→ linearly dependent cause 0̅ not only happens when x1=x2=x3=0

Note:

rank = number of column n → linearly independent, in another words no free var, R = , N(A) = 0

CLAIM:

Any set of n vectors in Rm (m x n) must be linearly dependent when n > m.

proof:

rank must < n , n - r ≠ 0 ,have free var.

Definition: spans a vector space

A set of vectors spans a vector space if their linear combinations fill the space.

Remark:

The column of a matrix spans its column space.

### Row space

Definition: row space

The row space of a mxn matrix is the subspace of Rn spanned by the rows of A.

row space of A: C(AT), column space of AT.

### Basis

Definition: basis

A basis of a vector space is a sequence of vectors satisfying two properties:

1. The basis vectors are linearly independent.
2. They span the vector space. (space中的任何向量可寫成basis的線性組合)

Example:

1. constitute a basis for R3.
2. constitute a basis for R3.

CLAIM:

Any n independent vectors in Rn are a basis for Rn.

CLAIM:

The columns of a nonsingular (invertible) matrix (nxn) is a basis for Rn.

Remark:

Rn has infinity different bases.

CLAIM:

There is one and only one way to describe any vector in a vector space with the linear combination of the vectors in one basis.

proof:

{v1, v2,…,vn} is a basis.

Suppose v̅ = a1v1 + a2v2 +…+ anvn

= b1v1 + b2v2 +…+ bnvn

and those vectors in combinations should be independent, therefore

0̅ = v̅ - v̅ = (a1-b1)v1 + (a2-b2)v2 + … +(an-bn)vn

and all scalar (an-bn) = 0 → an = bn

Example:

1. A = find the basis of column space C(A).

A =

any free column is the linear combination of pivot columns.

C(R) 的basis = R的pivot columns.

因為(Ax̅ = b̅ 的解) = (Rx̅ = b̅ 解)

C(A) = {, x1,x2 ∈ R }

{} is the basis of C(A)

Note:

C(R) ≠ C(A) 因為高斯消去時column改變了

CLAIM:

If Bv {v1, v2, …vn} & Bw {w1, w2, …wm} both the basis of vector space V, m=n.

proof:

Suppose m > n,

All w in Bw can be describe in wj = aj1v1+aj2v2+…+ ajnvn; j=1, 2, …m

b1w1 + b2w2 +…+ bmwm

=

1 x m m x n n x 1

since {v1, v2, …vn} is basis of V, it must be linearly independent, hence

m > n, b1, b2, …bn always has free var

→ w1, w2, …, wm is linearly dependent

→but this statement contradict Bw is basis (w1…wm are linearly independent).

→m > n can’t stand

so as n > m → m must = n.

### Dimension

Definition:

The dimension of a vector space is the number of vectors in every basis.

Example:

1. dim(Rn) = n
2. A = dim(C(A)) = ?

→R =

C(A) = {, x1,x2 ∈ R }

dim(C(A)) = 2 (=dim(C(R)))

1. M = the vector space of all 2 x 2 real matrices

basis of M:

dim(M) = 4

1. Dimension of n x n real matrix space = n2.

Dimension of the subspace of all n x n upper triangular matrices = (n+1) \*n/2.

(上底加下底乘高除2; subspace: x, y ∈ Z, x + y ∈ Z)

Dimension of the subspace of all n x n diagonal matrices = n.

Dimension of the subspace of all n x n symmetric matrices = (n+1) \*n/2.

Dimension of the subspace of all n x n unsymmetric matrices =

不對稱加不對稱不見得不對稱，Subspace不存在

#### Dimension of the four Subspace

Am x n

A =

→ R =

rank = 2,

1. row space C(AT): C(AT) is subspace of Rn

basis for C(AT) = pivot row =

dim(RT) = number of pivot row = rank = 2

since EA = R, A = E-1R, rows in A is a linear combination of rows in R, vice versa.

C(AT) = C(RT)

dim(C(AT)) = dim(RT) = rank (AT) = rank(A) = 2

1. column space C(A): C(A) is subspace of Rm

basis for C(A) = pivot column of A = (≠pivot column of R)

dim(C(A)) = rank(A) = 2

1. nullspace N(A): N(A) is subspace of Rn

N(A) = {x̅: Ax̅ = 0}

basis for N(A) = the vector of the homogeneous solution of A

N(A) ={}

dim(N(A)) =number of free var = n – r = 3

1. left nullspace N(AT): N(AT) is subspace of Rm

N(AT)= {y̅: ATy̅ = 0̅} = {y̅: y̅TA = 0̅T}

y̅TA = 0̅T ↔ y̅TR = 0̅T

, basis of N(AT) =

hence 0 row gives the free var, and the free var gives the degrees of freedom of N(AT) (pivot would make the relative var = 0)

dim(N(AT)) = number of 0 row = m – r = 1. (suppose y̅ must be 1 x m)

since EA = R

tell us

the basis to N(AT) = the row vector set of var that make A to a 0̅T

= (≠the row vector set of var that make R to a 0̅T)

NOTE:

Due to the Gaussian Elimination, the operate on the rows destroy the column’s structure of the original matrix, so the basis to the original matrix’s Column space & Left Nullspace (subspaces of R(number of element in column, number of row) )

doesn’t equal to the basis to the reduce echelon form’s(R) subspaces (of R(number of row)).

## Fundamental Theorem of Linear Algebra Part 1

* Column space and Row space both have dimension r.
* Nullspace and Left Nullspace have dimension n – r, m – r

# Orthogonality

# Eigenvalue and Eigenvectors

## Introduction to Eigenvectors

Example

In a certain town, 30% of the married women get divorced each year and 20% of the single women get married each year. suppose initially there are 8000 married women and 2000 single women remains constant.

w0 = , w1 = , wn = An w0

w20 ≒ , w100 ≒ ,← steady state

suppose x̅1 = then Ax̅1 =

x̅2 = then Ax̅2 = = 0.5w̅2

w0 =

w1 = Aw0 = 2000 Ax̅1 – 4000 Ax̅2 = 2000 x̅1 – 4000(0.5) x̅2

w2 = Aw1 = A(2000 x̅1 – 4000(0.5) x̅2 = 2000 x̅1 – 4000(0.5)2 x̅2

wn = 2000 x̅1 – 4000(0.5)n x̅2 ≒

Definition:

for A nxn, if Ax̅ = λx̅, for a nonzero(n x 1) vector x̅

λ = eigenvalue of A

x̅ = the associated eigenvector

Solve Ax̅ = λx̅, for a nonzero vector x̅

Ax̅ - λ x̅ = 0̅

→(A – λ I) x̅= 0̅ (λ is scalar)

x̅ is nonzero vector → N(A - λ I) ≠ 0

→ (A - λ I) is singular → det (A - λ I) = 0

Example:

1. A =

det(A - λ I) = = 0

= λ2 – 1.5λ+0.56 - 0.06

λ= 1, 1/2

λ= 1, x̅ = {x2}

λ= 0.5, x̅ = {x2}

eigenvector =

1. A =

det(A - λ I) = λ2 – λ+0.25 - 0.25 = 0

λ= 1, 0

λ= 1, x̅ = {x2}

λ= 0, x̅ = {x2}

for λ= 0, Ax̅ = λx̅ → Ax̅ = 0̅ → A is singular.

1. Q =

Q, -90 rotation

det(Q –λ I) = 0, λ = +i, -i.

no real eigenvalue, hence there are no real eigenvector.

since Q is a rotation by -90 degree, no vector Qx̅ stay in the same direction as x̅.

複數根必成對出現(共厄複數)，所以單數次必有1實根

CLAIM:

λ is eigenvalue of A, λ2 is eigenvalue of A2, and eigenvector stay the same.

proof:

Ax̅ = λ x̅, A2x̅ = λ Ax̅ = λ (λ x̅ ) = λ2 x̅

Example:

1. A = , λ = 1, 0.5

λ1 + λ2 = 1.5

λ1 λ2 = 0.5

det(A) = 0.56 - 0.06 = 0.5

0.7 + 0.8 = 1.5, trace(A) = 1.5 = 對角線相加

1. A = , λ = 1, 0

λ1 + λ2 = 1

λ1 λ2 = 0

det(A) = 0

trace(A) = 1

CLAIM:

λ1 \* λ2 \* … \* λn = det(A)

det(A) = ±pivot1 \* pivot2 \* … \* pivotn

proof:

det(A – λ I) =

= (λ1 – λ) (λ2 – λ) … (λn – λ)

λ = 0, dea(A) = λ1 \* λ2 \* … \* λn

CLAIM:

λ1 + λ2 + … + λn = trace(A)

proof:

(λ1 – λ) (λ2 – λ) … (λn – λ), 能夠有(λ1 + λ2 + … + λn)係數的時候為λn-1, det(A - λ I) 中包含有λn-1 項的只有對角線相乘

(a11 – λ) (a22 – λ) … (ann – λ)

(-1)n-1(a11+a22+…+ann) = (-1)n-1(λ1 + λ2 +…+ λn)

(a11+a22+…+ann) = (λ1 + λ2 +…+ λn)

S-1AS = diagonal (λ) =

A = S diagonal (λ) S-1